Fundamentals of Magnetic Island Theory in Tokamaks

Richard Fitzpatrick

Institute for Fusion Studies

University of Texas at Austin

Austin, TX, USA

Talk available at

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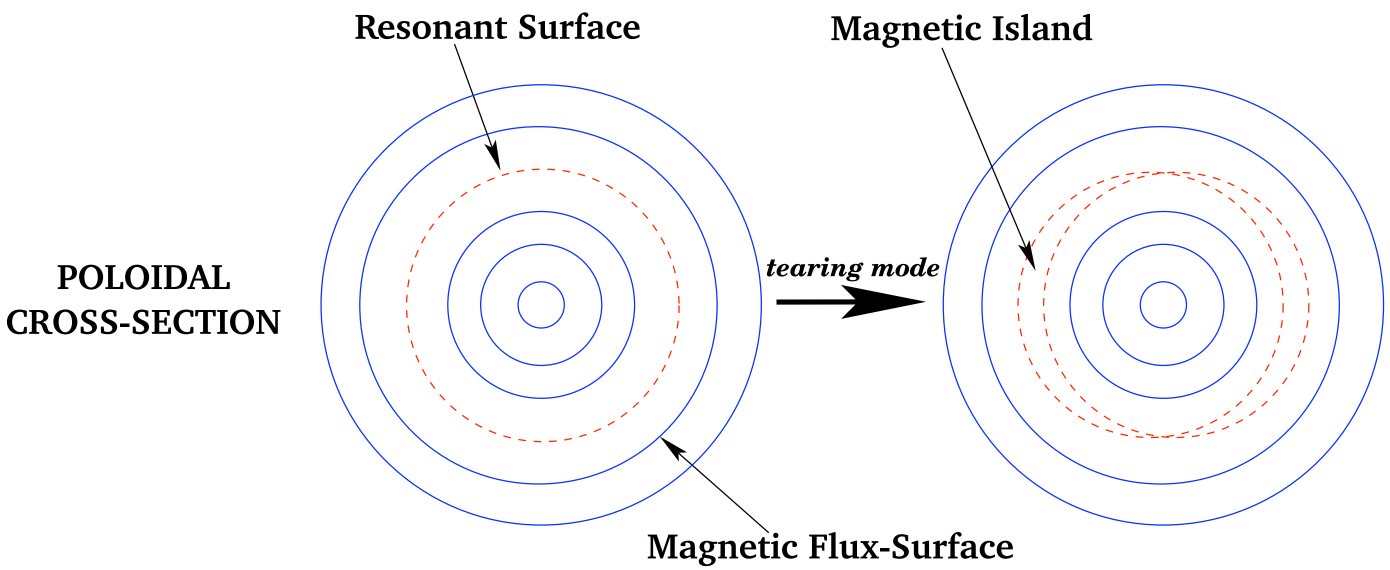
Macroscopic Instabilities

- Two main types of macroscopic instabilities in tokamaks:[a]
  - Catastrophic "ideal" (*i.e.*, non-reconnecting) instabilities, which disrupt plasma in few micro-seconds. Can be avoided by limiting plasma pressure and current.
  - Slowly growing "tearing" instabilities, which reconnect magnetic flux-surfaces to form *magnetic islands*, thereby degrading their confinement properties. Much harder to avoid.

[a] *MHD Instabilities*, G. Bateman (MIT, 1978).

Magnetic Islands



- Helical structures, centered on *rational magnetic flux-surfaces* which satisfy $\vec{k} \cdot \vec{B} = 0$, where $\vec{k}$ is wavenumber of mode, and $\vec{B}$ is equilibrium magnetic field.
- Effectively "short-circuit" confinement by allowing heat/particles to radially transit island region by rapidly flowing along magnetic field-lines, rather than slowly diffusing across flux-surfaces.

Need for Magnetic Island Theory

- Magnetic island formation associated with *nonlinear* phase of tearing mode growth (*i.e.*, when radial island width becomes greater than linear layer width at rational surface).
- In very hot plasmas found in modern-day tokamaks, linear layers so thin that tearing mode already in nonlinear regime when first detected.
- Linear tearing mode theory largely irrelevant. Require nonlinear magnetic island theory to explain experimental observations.

MHD Theory

- Tearing modes are macroscopic instabilities which affect whole plasma. Natural to investigate them using some form of *fluid-theory*.

- Simplest fluid theory is well-known *magnetohydrodynamical approximation*,[a] which effectively treats plasma as *single-fluid*.

- Shall also use *slab approximation* to simplify analysis.

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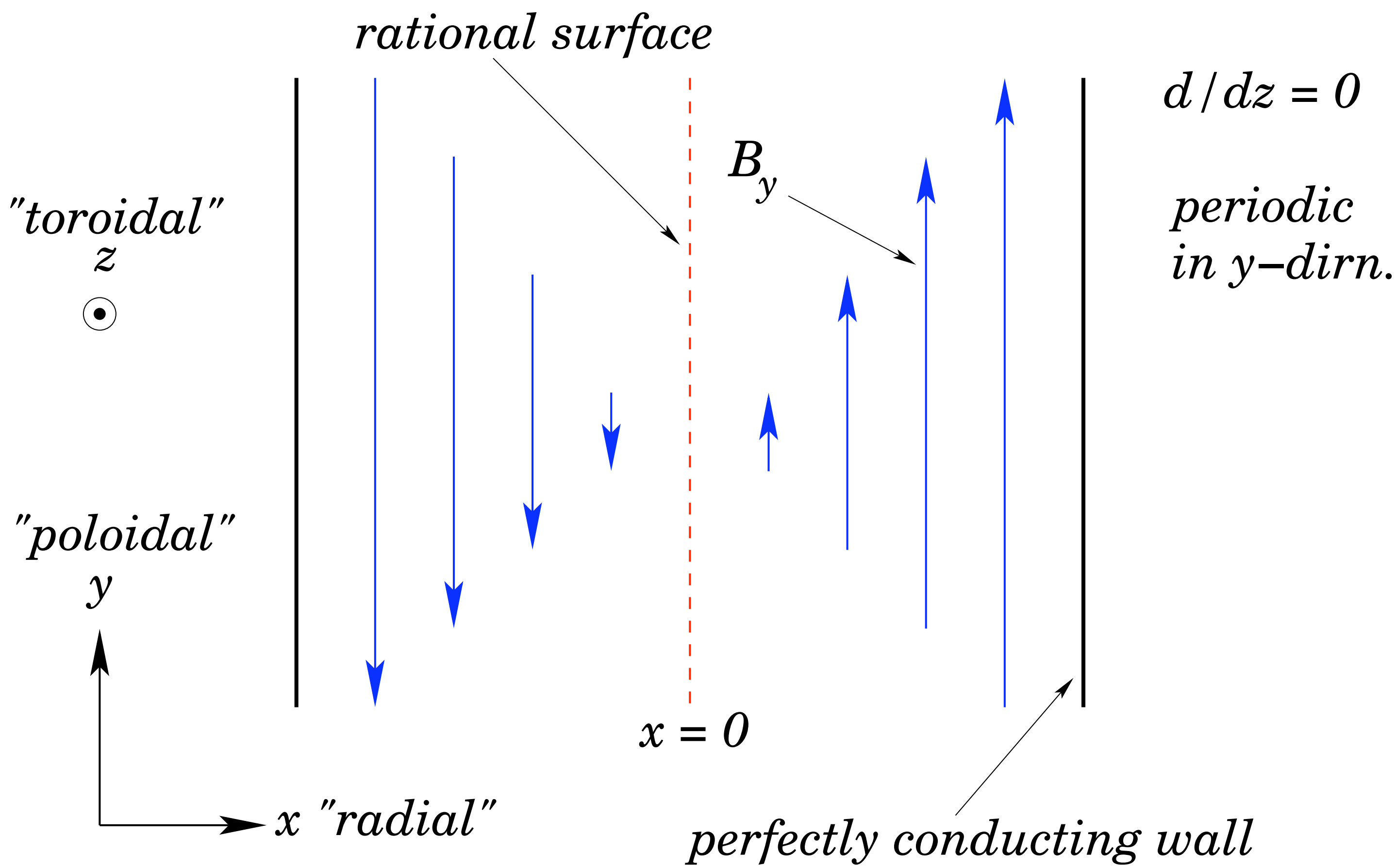
[a] *Plasma Confinement*, R.D. Hazeltine, and J.D. Meiss (Dover, 2003).

Slab Approximation



Slab Model

- Cartesian coordinates: (x, y, z). Let $\partial/\partial z \equiv 0$.
- Assume presence of dominant uniform "toroidal" $\vec{B}_z\,\vec{z}$.
- All field-strengths normalized to B_z.
- All lengths normalized to equilibrium magnetic shear-length:

$$L_s = B_z/B_y'(0).$$

- All times normalized to shear-Alfvén time calculated with B_z.
- Perfect wall boundary conditions at $x = \pm a$.
- Wavenumber of tearing instability: $\vec{k} = (0, k, 0)$, so $\vec{k}\cdot\vec{B} = 0$ at $x = 0$. Hence, rational surface at $x = 0$.

Model MHD equations

- Let $\vec{B}_\perp = \nabla\psi \times \vec{z}$ and $\vec{V} = \nabla\phi \times \vec{z}$, where $\vec{V}$ is $\vec{E} \times \vec{B}$ velocity.
- $\vec{B} \cdot \nabla\psi = \vec{V} \cdot \nabla\phi = 0$, so ψ maps magnetic flux-surfaces, and ϕ maps stream-lines of $\vec{E} \times \vec{B}$ fluid.
- Incompressible MHD equations:[a]

$$
\begin{aligned}
\frac{\partial\psi}{\partial t} &= [\phi, \psi] + \eta\, J, \\
\frac{\partial U}{\partial t} &= [\phi, U] + [J, \psi] + \mu\, \nabla^2 U,
\end{aligned}
$$

  where $J = \nabla^2\psi$, $U = \nabla^2\phi$, and $[A, B] = A_x\, B_y - A_y\, B_x$. Here, η is resistivity, and μ is viscosity. In normalized units: $\eta, \mu \ll 1$.

- First equation is z-component of Ohm's law. Second equation is z-component of curl of plasma equation of motion.

[a] *Plasma Confinement*, R.D. Hazeltine, and J.D. Meiss (Dover, 2003).

Outer Region

- In "outer region", which comprises most of plasma, non-linear, non-ideal (η and μ), and inertial ($\partial/\partial t$ and $\vec{V}\cdot\nabla$) effects negligible.

- Vorticity equation reduces to

$$[J, \psi] \simeq 0.$$

- When linearized, obtain $\psi(x, y) = \psi^{(0)}(x) + \psi^{(1)}(x)\,\cos(k\,y)$, where $B_y^{(0)} = -d\psi^{(0)}/dx$, and

$$\left(\frac{d^2}{dx^2} - k^2\right)\psi^{(1)} - \left(\frac{d^2B_y^{(0)}/dx^2}{B_y^{(0)}}\right)\psi^{(1)} = 0.$$

- Equation is *singular* at rational surface, $x = 0$, where $B_y^{(0)} = 0$.

Tearing Stability Index

- Find tearing eigenfunction, $\psi^{(1)}(x)$, which is continuous, has tearing parity $[\psi^{(1)}(-x) = \psi^{(1)}(x)]$, and satisfies boundary condition $\psi^{(1)}(a) = 0$ at conducting wall.

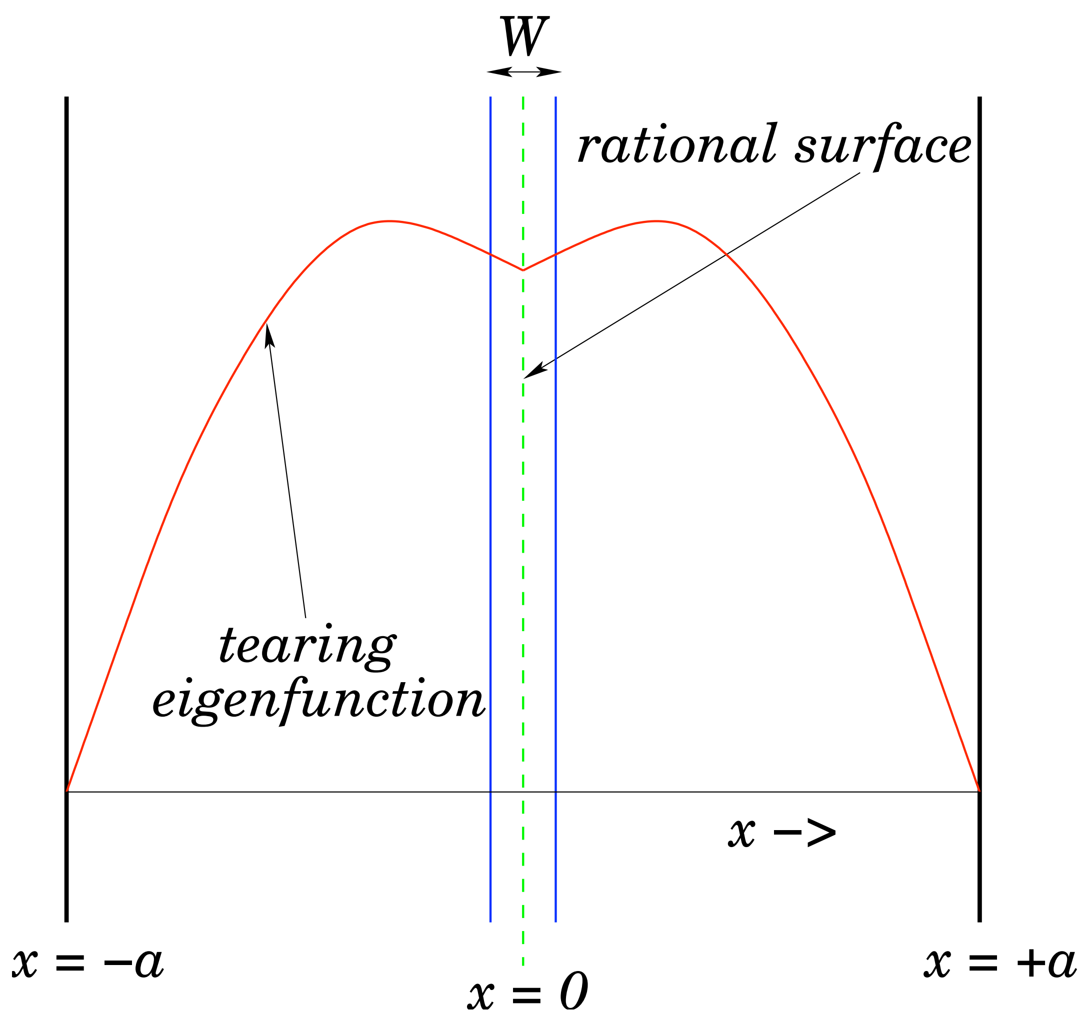
- In general, eigenfunction has *gradient discontinuity* across rational surface (at $x = 0$). Allowed because tearing mode equation singular at rational surface.

- Tearing stability index:

$$\Delta' = \left[\frac{d\ln\psi^{(1)}}{dx}\right]_{0-}^{0+}.$$

- According to conventional MHD theory,[a] tearing mode is unstable if $\Delta' > 0$.

[a] H.P. Furth, J. Killeen, and M.N. Rosenbluth, Phys. Fluids **6**, 459 (1963).


W
rational surface
tearing
eigenfunction
x ->
x = −a
x = 0
x = +a

Inner Region

- "Inner region" centered on rational surface, $x = 0$. Of extent, $W \ll 1$, where W is magnetic island width (in x).
- In inner region, non-ideal effects, non-linear effects, and plasma inertia can all be important.
- Inner solution must be asymptotically matched to outer solution already obtained.

Constant-ψ Approximation

- $\psi^{(1)}(x)$ generally does not vary significantly in x over inner region:

$$|\psi^{(1)}(W) - \psi^{(1)}(0)| \ll |\psi^{1}(0)|.$$

- *Constant-ψ approximation*: treat $\psi^{(1)}(x)$ as constant in x over inner region.

- Approximation valid provided

$$|\Delta'|\, W \ll 1,$$

which is easily satisfied for conventional tearing modes.

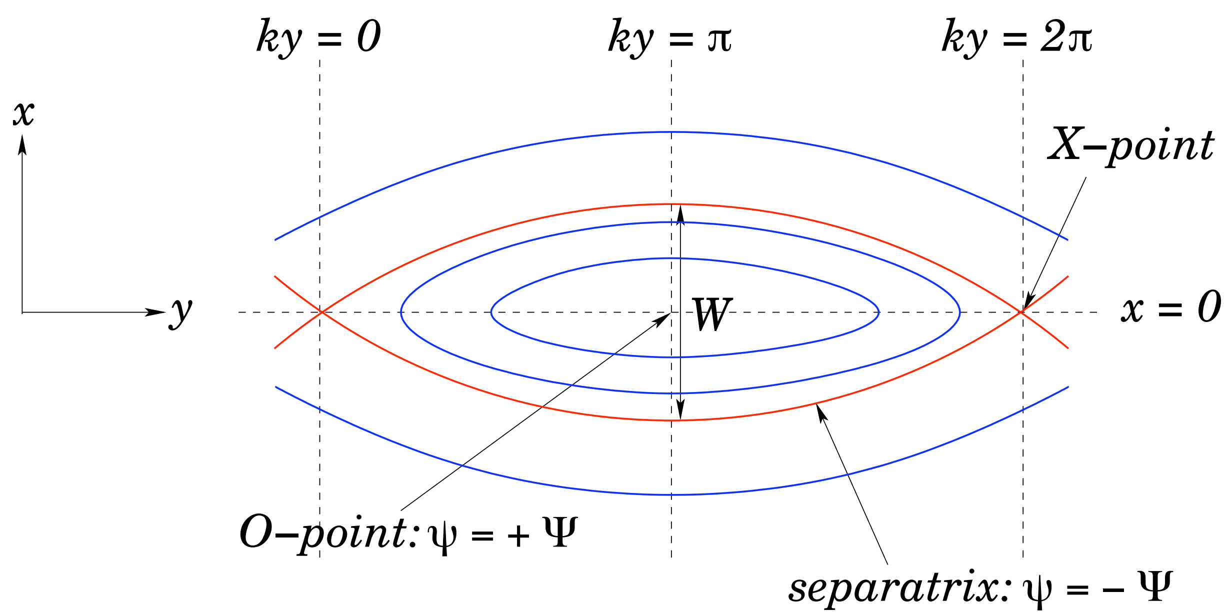
Constant-ψ Magnetic Island

- In vicinity of rational surface, $\psi^{(0)} \to -x^2/2$, so

$$\psi(x, y, t) \simeq -x^2/2 + \Psi(t)\,\cos\theta,$$

where $\Psi = \psi^{(1)}(0)$ is "reconnected flux", and $\theta = k\,y$.

- Full island width, $W = 4\sqrt{\Psi}$.



Flux-Surface Average Operator

- Flux-surface average operator is annihilator of Poisson bracket $[A, \psi] \equiv \vec{B} \cdot \nabla A \equiv k\, x\, (\partial A/\partial \theta)_\psi$ for any A: *i.e.*,

$$\langle [A, \psi] \rangle \equiv 0.$$

- Outside separatrix:

$$\langle f(\psi, \theta) \rangle = \oint \frac{f(\psi, \theta)}{|x|} \frac{d\theta}{2\pi}.$$

- Inside separatrix:

$$\langle f(s, \psi, \theta) \rangle = \int_{-\theta_0}^{\theta_0} \frac{f(s, \psi, \theta) + f(-s, \psi, \theta)}{2\,|x|} \frac{d\theta}{2\pi},$$

where $s = \mathrm{sgn}(x)$, and $x(s, \psi, \theta_0) = 0$.

MHD Flow -I

- Move to island frame. Look for steady-state solution: $\partial/\partial t = 0$.[a]
- Ohm's law:

$$0 \simeq [\phi, \psi] + \eta J.$$

- Since $\eta \ll 1$, first term potentially much larger than second.
- To lowest order:

$$[\phi, \psi] \simeq 0.$$

- Follows that

$$\phi = \phi(\psi):$$

i.e., MHD flow constrained to be around flux-surfaces.

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[a] F.L. Waelbroeck, and R. Fitzpatrick, Phys. Rev. Lett. **78**, 1703 (1997).

MHD Flow - II

- Let

$$M(\psi) = \frac{d\phi}{d\psi}.$$

- Easily shown that

$$V_y = x\,M.$$

- By symmetry, $M(\psi)$ is *odd* function of x. Hence,

$$M = 0$$

inside separatrix: *i.e.*, no flow inside separatrix in island frame. Plasma *trapped* within magnetic separatrix.

MHD Flow - III

- Vorticity equation:

$$0 \simeq [-M\,U + J, \psi] + \mu\,\nabla^4\phi.$$

- Flux-surface average, recalling that $\langle [A, \psi]\rangle = 0$:

$$\langle \nabla^4\phi\rangle \equiv -\frac{d^2}{d\psi^2}\left(\langle x^4\rangle\,\frac{dM}{d\psi}\right) \simeq 0.$$

- Solution outside separatrix:

$$M(\psi) = \mathrm{sgn}(x)\,M_0\,\int_{-\Psi}^{\psi} d\psi/\langle x^4\rangle \bigg/ \int_{-\Psi}^{-\infty} d\psi/\langle x^4\rangle.$$

MHD Flow - IV

- Note that

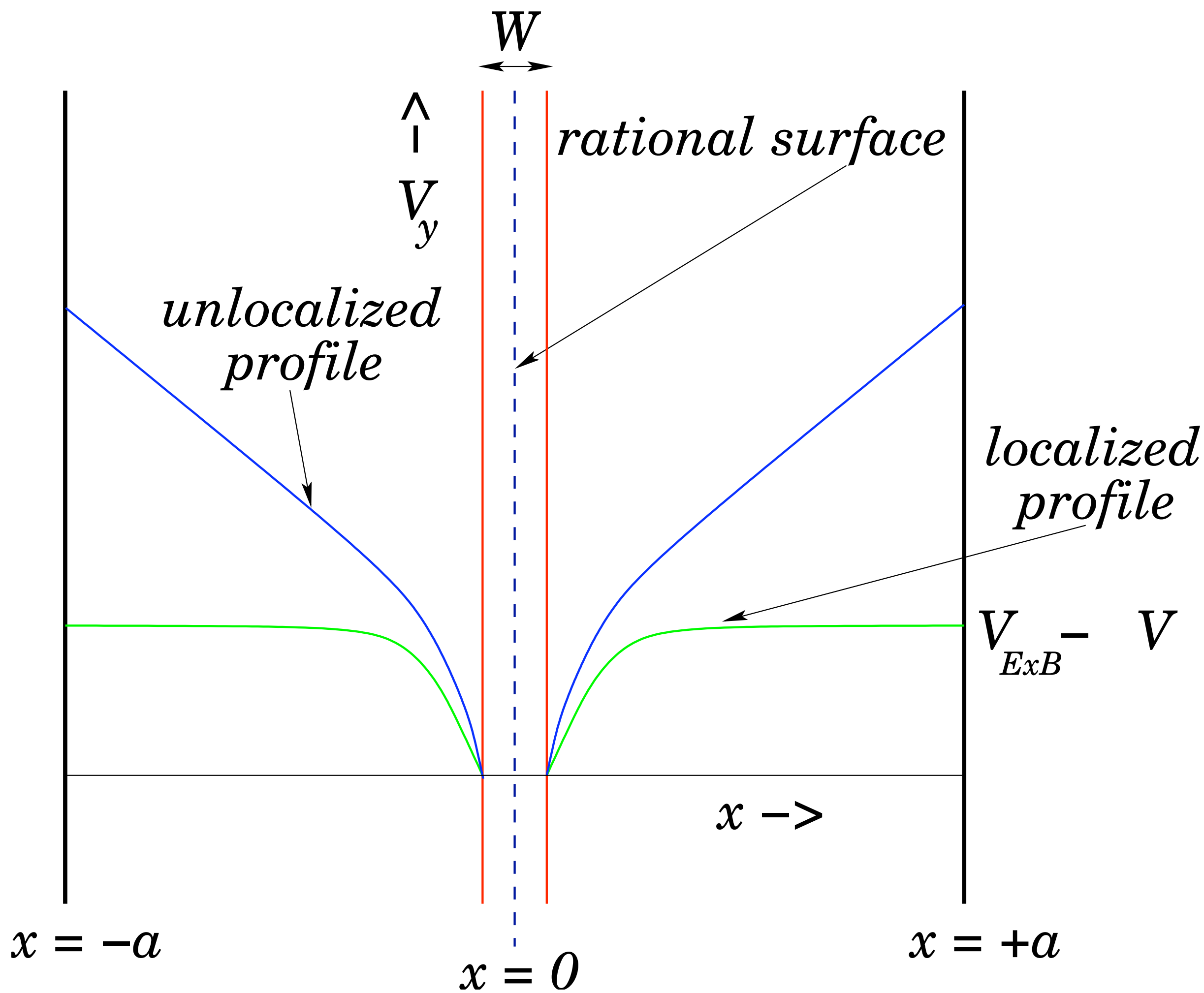
$$V_y = x\,M \to |x|\,M_0$$

  as $|x|/W \to \infty$.

- V-shaped velocity profile which extends over whole plasma.

- Expect *isolated* magnetic island to have *localized* velocity profile. Suggests that $M_0 = 0$ for isolated island.

- Hence, zero MHD flow in island frame: *i.e.*, island propagates at local $\vec{E} \times \vec{B}$ velocity.


W
V_y
rational surface
unlocalized profile
localized profile
$V_{ExB} - V$
x ->
x = −a
x = 0
x = +a

Rutherford Equation - I

- Asymptotic matching between inner and outer regions yields:

$$\Delta' \Psi = -4 \int_{+\Psi}^{-\infty} \langle J \cos\theta \rangle \, d\psi.$$

- In island frame, in absence of MHD flow, vorticity equation reduces to

$$[J, \psi] \simeq 0.$$

- Hence,

$$J = J(\psi).$$

Rutherford Equation - II

- Ohm's law:

$$\frac{d\Psi}{dt}\cos\theta \simeq [\phi,\psi] + \eta\, J(\psi).$$

- Have shown there is no MHD-flow [*i.e.*, $\phi \sim O(1)$], but can still be *resistive flow* [*i.e.*, $\phi \sim O(\eta)$].

- Eliminate resistive flow by flux-surface averaging:

$$\frac{d\Psi}{dt}\langle\cos\theta\rangle \simeq \eta\, J(\psi)\,\langle 1\rangle.$$

- Hence,

$$\Delta'\,\Psi \simeq -\frac{4}{\eta}\frac{d\Psi}{dt}\int_{+\Psi}^{-\infty}\frac{\langle\cos\theta\rangle^2}{\langle 1\rangle}\,d\psi.$$

Rutherford Equation - III

- Use $W = 4\sqrt{\Psi}$, and evaluate integral. Obtain *Rutherford island width evolution equation*:[a]

$$\frac{0.823}{\eta}\frac{dW}{dt} \simeq \Delta'.$$

- According to Rutherford equation, island grows *algebraically* on *resistive time-scale*.

- Rutherford equation does not predict island saturation.

[a] P.H. Rutherford, Phys. Fluids **16**, 1903 (1973).

Rutherford Equation - IV

- Higher order asymptotic matching between inner and outer regions yields: [a]

$$\frac{0.823}{\eta}\frac{dW}{dt} \simeq \Delta' - 0.41 \left(- \frac{d^4 B_y^{(0)}/dx^4}{d^2 B_y^{(0)}/dx^2} \right)_{x=0} W.$$

- Hence, saturated $(d/dt = 0)$ island width is

$$W_0 = \frac{\Delta'}{0.41} \left(- \frac{d^2 B_y^{(0)}/dx^2}{d^4 B_y^{(0)}/dx^4} \right)_{x=0}.$$

[a] F. Militello, and F. Porcelli, Phys. Plasmas **11**, L13 (2004). D.F. Escande, and M. Ottaviani, Physics Lett. A **323**, 278 (2004).

MHD Theory: Summary

- Tearing mode unstable if $\Delta' > 0$.
- Island propagates at local $\vec{E} \times \vec{B}$ velocity at rational surface.
- Island grows algebraically on resistive time-scale.
- Saturated island width:

$$W_0 = \frac{\Delta'}{0.41} \left(-\frac{d^2 B_y^{(0)}/dx^2}{d^4 B_y^{(0)}/dx^4} \right)_{x=0}.$$

Drift-MHD Theory

- In drift-MHD approximation, analysis retains *charged particle drift velocities*, in addition to $\vec{E} \times \vec{B}$ velocity.
- Essentially two-fluid theory of plasma.
- Characteristic length-scale, ρ, is *ion Larmor radius calculated with electron temperature*.
- Characteristic velocity is diamagnetic velocity, V_*, where

$$n\,e\,\vec{V}_* \times \vec{B} = \nabla P.$$

- Normalize all lengths to ρ, and all velocities to V_*.

Basic Assumptions

- Retain slab model, for sake of simplicity.
- Assume parallel electron heat transport sufficiently strong that $T_e = T_e(\psi)$.
- Assume $T_i/T_e = \tau = \mathrm{constant}$, for sake of simplicity.

Basic Definitions

- Variables:
  - ψ - magnetic flux-function.
  - J - parallel current.
  - ϕ - guiding-center (*i.e.*, MHD) stream-function.
  - U - parallel ion vorticity.
  - n - electron number density (minus uniform background).
  - V_z - parallel ion velocity.
- Parameters:
  - $\alpha = (L_n/L_s)^2$, where L_n is equilibrium density gradient scale-length.
  - η - resistivity. D - (perpendicular) particle diffusivity. $\mu_{i/e}$ - (perpendicular) ion/electron viscosity.

Drift-MHD Equations - I

- Steady-state drift-MHD equations:[a]

$$
\begin{aligned}
\psi &= -x^2/2 + \Psi \cos\theta, \qquad U = \nabla^2 \phi, \\
0 &= [\phi - n, \psi] + \eta\, J, \\
0 &= [\phi, U] - \frac{\tau}{2}\left\{\nabla^2[\phi, n] + [U, n] + [\nabla^2 n, \phi]\right\} \\
&\quad + [J, \psi] + \mu_i \nabla^4(\phi + \tau\, n) + \mu_e\, \nabla^4(\phi - n), \\
0 &= [\phi, n] + [V_z + J, \psi] + D\, \nabla^2 n, \\
0 &= [\phi, V_z] + \alpha\, [n, \psi] + \mu_i\, \nabla^2 V_z.
\end{aligned}
$$

[a] R.D. Hazeltine, M. Kotschenreuther, and P.J. Morrison, Phys. Fluids **28**, 2466 (1985).

Drift-MHD Equations - II

- Symmetry: ψ, J, V_z even in x. ϕ, n, U odd in x.
- Boundary conditions as $|x|/W \to \infty$:
  - $n \to -(1+\tau)^{-1}\,x$.
  - $\phi \to -V\,x$.
  - $J, U, V_z \to 0$.
- Here, V is island phase-velocity in $\vec{E}\times\vec{B}$ frame.
- $V = 1$ corresponds to island propagating with electron fluid. $V = -\tau$ corresponds to island propagating with ion fluid.
- Expect

$$1 \gg \alpha \gg \eta, D, \mu_i, \mu_e.$$

Electron Fluid

- Ohm's law:

$$0 = [\phi - n, \psi] + \eta\, J.$$

- Since $\eta \ll 1$, first term potentially much larger than second.

- To lowest order:

$$[\phi - n, \psi] \simeq 0.$$

- Follows that

$$n = \phi + H(\psi):$$

  i.e., electron stream-function $\phi_e = \phi - n$ is *flux-surface function*. Electron fluid flow constrained to be around flux-surfaces.

Sound Waves

- Parallel flow equation:

$$0 = [\phi, V_z] + \alpha\,[n, \psi] + \mu_i\,\nabla^2 V_z.$$

- Highlighted term dominant provided

$$W \gg \alpha^{-1/2} = L_s/L_n.$$

- If this is case then to lowest order

$$n = n(\psi),$$

  which implies $n = 0$ inside separatrix.

- So, if island sufficiently wide, *sound-waves* able to *flatten density profile* inside island separatrix.

Subsonic vs. Supersonic Islands

- Wide islands satisfying

$$W \gg L_s/L_n$$

termed *subsonic* islands. Expect such islands to exhibit flattened density profile within separatrix. Subsonic islands strongly coupled to both electron and ion fluids.

- Narrow islands satisfying

$$W \ll L_s/L_n$$

termed *supersonic* islands. No flattening of density profile within separatrix. Supersonic islands strongly coupled to electron fluid, but only weakly coupled to ion fluid.

Subsonic Islands [a]

- To lowest order:

$$\phi = \phi(\psi), \;\; n = n(\psi).$$

- Follows that both electron stream-function, $\phi_e = \phi - n$, and ion stream-function, $\phi_i = \phi + \tau\, n$, are flux-surface functions. Both electron and ion fluid flow constrained to follow flux-surfaces.

- Let

$$M(\psi) = d\phi/d\psi, \;\; L(\psi) = dn/d\psi.$$

- Follows that

$$V_{E\times B\,y} = x\, M, \;\; V_{e\,y} = x\,(M - L), \;\; V_{i\,y} = x\,(M + \tau\, L).$$

[a] R. Fitzpatrick, F.L. Waelbroeck, Phys. Plasmas **12**, 022307 (2005).

Density Flattening

- By symmetry, both $M(\psi)$ and $L(\psi)$ are *odd* functions of x. Hence,

$$M(\psi) = L(\psi) = 0$$

inside separatrix: *i.e.*, no electron/ion flow within separatrix in island frame.

- Electron/ion fluids constrained to propagate with island inside separatrix.

- Density profile *flattened* within separatrix.

Analysis - I

- Density equation reduces to

$$0 \simeq [V_z + J, \psi] + D\,\nabla^2 n.$$

- Vorticity equation reduces to

$$\begin{aligned} 0 &\simeq \left[-M\,U - (\tau/2)(L\,U + M\,\nabla^2 n) + J, \psi\right] \\ &\quad + \mu_i\,\nabla^4(\phi + \tau\,n) + \mu_e\,\nabla^4(\phi - n). \end{aligned}$$

- Flux-surface average both equations, recalling that $\langle [A, \psi] \rangle = 0$.

Analysis - II

- Obtain

$$\langle \nabla^2 n \rangle \simeq 0,$$

and

$$(\mu_i + \mu_e)\,\langle \nabla^4 \phi \rangle + (\mu_i\,\tau - \mu_e)\,\langle \nabla^4 n \rangle \simeq 0.$$

- Solution outside separatrix:

$$M(\psi) = -\frac{(\mu_i\,\tau - \mu_e)}{(\mu_i + \mu_e)}\,L(\psi) + F(\psi),$$

where

$$L(\psi) = -\mathrm{sgn}(x)\,L_0/\langle x^2 \rangle,$$

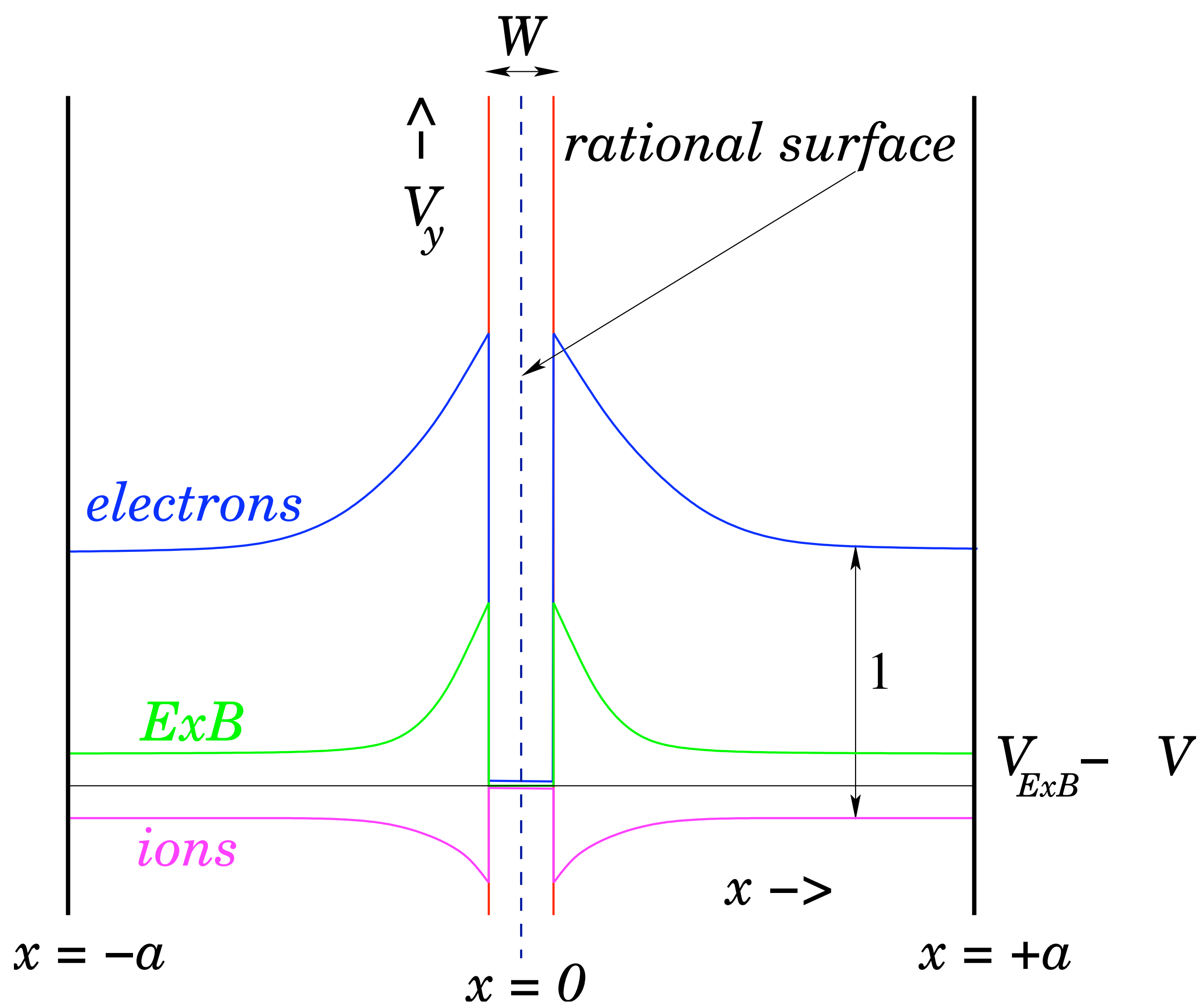
and $F(\psi)$ is previously obtained MHD profile:

$$F(\psi) = \mathrm{sgn}(x)\,F_0\,\int_{-\Psi}^{\psi} d\psi/\langle x^4 \rangle \Bigg/ \int_{-\Psi}^{-\infty} d\psi/\langle x^4 \rangle.$$

Velocity Profiles

- As $|x|/W \to \infty$ then $x\,L \to L_0$ and $x\,F \to |x|\,F_0$.
- $L(\psi)$ corresponds to *localized* velocity profile. $F(\psi)$ corresponds to *non-localized* profile. Require localized profile, so $F_0 = 0$.
- Velocity profiles outside separatrix (using b.c. on n):

$$
\begin{aligned}
V_{y\,i} &\simeq +\frac{\mu_e}{\mu_i + \mu_e}\,\frac{|x|}{\langle x^2\rangle},\\
V_{y\,E\times B} &\simeq -\frac{(\mu_i\,\tau - \mu_e)}{(1+\tau)\,(\mu_i + \mu_e)}\,\frac{|x|}{\langle x^2\rangle},\\
V_{y\,e} &= -\frac{\mu_i}{\mu_i + \mu_e}\,\frac{|x|}{\langle x^2\rangle}.
\end{aligned}
$$


W
^
V_y
rational surface
electrons
1
ExB
$V_{ExB} - V$
ions
x ->
x = −a
x = 0
x = +a

Island Propagation

- As $|x|/W \to \infty$ expect $V_{y\,E\times B} \to V_{EB} - V$, where V_{EB} is unperturbed (*i.e.*, no island) $\vec{E} \times \vec{B}$ velocity at rational surface (in lab. frame), and V is island phase-velocity (in lab. frame).

- Hence
$$V = V_{EB} + \frac{(\mu_i\,\tau - \mu_e)}{(1+\tau)\,(\mu_i + \mu_e)}.$$

- But unperturbed ion/electron fluid velocities (in lab. frame):
$$V_i = V_{EB} + \tau/(1+\tau), \quad V_e = V_{EB} - 1/(1+\tau).$$

- Hence
$$V = \frac{\mu_i}{\mu_i + \mu_e}\,V_i + \frac{\mu_e}{\mu_i + \mu_e}\,V_e.$$
So, island phase-velocity is *viscosity weighted average* of unperturbed ion/electron fluid velocities.

Polarization Term - I

- Vorticity equation yields

$$J_c \simeq \frac{1}{2}\left(x^2 - \frac{\langle x^2\rangle}{\langle 1\rangle}\right)\frac{d[M\,(M+\tau\,L)]}{d\psi} + I(\psi)$$

outside separatrix, where J_c is part of J with $\cos\theta$ symmetry.

- As before, flux-surface average of Ohm's law yields:

$$\langle J_c\rangle = I(\psi)\langle 1\rangle = \eta^{-1}\,\frac{d\Psi}{dt}\langle\cos\theta\rangle.$$

- Hence

$$J_c \simeq \frac{1}{2}\left(x^2 - \frac{\langle x^2\rangle}{\langle 1\rangle}\right)\frac{d[M\,(M+\tau\,L)]}{d\psi} + \eta^{-1}\,\frac{d\Psi}{dt}\,\frac{\langle\cos\theta\rangle}{\langle 1\rangle}.$$

Polarization Term - II

- Asymptotic matching between inner and outer regions yields:

$$\Delta'\,\Psi = -4\int_{+\Psi}^{-\infty}\langle J_c\;\cos\theta\rangle\,d\psi.$$

- Evaluating flux-surface integrals, making use of previous solutions for M and L, obtain modified Rutherford equation:

$$\frac{0.823}{\eta}\frac{dW}{dt}\simeq\Delta'+1.38\,\beta\,\frac{(V-V_{EB})\,(V-V_i)}{(W/4)^3}.$$

- New term is due to *polarization current* associated with ion fluid flow around curved island flux-surfaces (in island frame). Obviously, new term is zero if island propagates with ion fluid: *i.e.*, $V=V_i$.

Drift-MHD Theory: Summary

- Results limited to large islands: *i.e.*, large enough for sound waves to flatten density profile.
- Island propagates at (perpendicular) viscosity weighted average of unperturbed (no island) ion and electron fluid velocities.
- Bootstrap term in Rutherford equation is destabiizing.
- Polarization term in Rutherford equation is stabilizing provided ion (perpendicular) viscosity greatly exceeds electron (perpendicular) viscosity (which is what we expect), and destabilizing otherwise.